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END INVARIANTS OF $SL(2,\mathbb{C})$-CHARACTERS OF THE ONCE-PUNCTURED TORUS ASSOCIATED WITH 2-BRIDGE LINKS

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1. Introduction

By extending the concept of a geometrically infinite end of a Kleinian group, Bowditch [3] introduced the notion of the end invariants of a type-preserving $SL(2,\mathbb{C})$-representation of the fundamental group $\pi_1(\boldsymbol{T})$ of the once-punctured torus $\boldsymbol{T}$. Tan, Wong and Zhang [14, 15] extended this notion (with slight modification) to an arbitrary $SL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. The purpose of this note is to explain the idea of the end invariants and to announce a result obtained in [9] which explicitly describes the sets of end invariants of the $SL(2,\mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of a hyperbolic 2-bridge link (Theorem 4.1).

2. Thurston's end invariants of punctured torus Kleinian groups

In this section, we recall the definition of Thurston's end invariants of punctured torus Kleinian groups, following [10, 11], and recall the classification theorem of punctured torus Kleinian groups due to Minsky [11].

Let $\rho : \pi_1(\boldsymbol{T}) \to PSL(2,\mathbb{C})$ be a faithful discrete representation, which is *type-preserving*, i.e., the image of conjugacy class associated to the boundary, $\rho(\partial\boldsymbol{T})$, is parabolic. The image $\Gamma := \rho(\pi_1(\boldsymbol{T}))$ is a free Kleinian group, and Γ together with its marking ρ is called a *punctured torus Kleinian group* or simply a *punctured torus group*. Let $M = \mathbb{H}^3/\Gamma$ be the quotient hyperbolic manifold and let P be the rank 1 cusp corresponding to $\rho(\partial\boldsymbol{T})$. Then P is homeomorphic to a product of an open annulus with the interval $(0,\infty)$. By [2], M is homeomorphic to $\boldsymbol{T}\times\mathbb{R}$, and the non-cuspidal part $\check{M} := M - P$ is homeomorphic to $\boldsymbol{T}_0\times\mathbb{R}$, where $\boldsymbol{T}_0$ is $\boldsymbol{T}$ minus an open neighborhood of the puncture. Thus $\check{M}$ has two ends e_- and e_+. To be precise, $\check{M}$ is identified with $\boldsymbol{T}_0\times(-1,1)\subset\boldsymbol{T}_0\times[-1,1]$, and e_+ denotes the end of $\check{M}$ whose neighborhoods are neighborhoods of $\boldsymbol{T}_0\times\{1\}$, and e_- the other end.

Let Ω be the (possibly empty) domain of discontinuity of Γ, and let $\overline{M}$ be the quotient $(\mathbb{H}^3\cup\Omega)/\Gamma$. Note that Ω/Γ is divided into two (possibly empty) pieces Ω_+/Γ and Ω_-/Γ corresponding to the ends e_+ and e_- (where $\Omega_\pm$ are the corresponding Γ-invariant subsets of Ω). There are three possibilities for each of the ends e_ϵ ($\epsilon\in\{+,-\}$), corresponding to three types of the *end invariant* $\nu_\epsilon(\rho)$ of the end e_ϵ:

(1) Ω_ϵ is a topological disk, and Ω_ϵ/Γ is a punctured torus. This determines a point in the Teichmüller space, $\mathcal{T}(\boldsymbol{T})$, of $\boldsymbol{T}$, i.e., the space of conformal structures on $\boldsymbol{T}$ modulo isotopy. The end invariant $\nu_\epsilon(\rho)\in\mathcal{T}(\boldsymbol{T})$ is defined to be the point.
(2) Ω_ϵ is an infinite union of round disks, and Ω_ϵ/Γ is a trice-punctured sphere, obtained from the boundary component $\boldsymbol{T}\times\{\epsilon 1\}$ by removing a simple closed curve

γ_ϵ. In this case the end invariant $\nu_\epsilon(\rho) \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ is defined to be the slope of γ_ϵ. It should be noted that the conjugacy class $\rho(\gamma_\epsilon)$ is parabolic.

(3) Ω_ϵ is empty. In this case the end invariant $\nu_\epsilon(\rho) \in \mathbb{R} - \mathbb{Q}$ is defined as follows. The condition $\Omega_\epsilon = \emptyset$ implies the existence of an infinite sequence, $\{\gamma_n\}$, of essential simple loops on $\boldsymbol{T}$, such that the geodesic representatives γ_n^* are eventually contained in any neighborhood of e_ϵ (see [2, 16]). Moreover the slope of γ_n converges in $\mathbb{R}$ to a unique irrational number. The end invariant $\nu_\epsilon(\rho)$ is defined to be this limiting irrational number.

In the first two cases, the end e_ϵ is said to be *geometrically finite*, whereas it is said to be *geometrically infinite* in the last case. In the last case, the end invariant is also called the *ending lamination* of the end.

Example 2.1. Let A be a matrix in $SL(2,\mathbb{Z})$ with $|\mathrm{tr}A| > 2$, and let φ_A be the self-homeomorphism of $\boldsymbol{T}$ induced by A. Let M_A be the punctured torus bundle with monodromy A, i.e.,

$$M_A = \boldsymbol{T} \times \mathbb{R}/(x,t) \sim (\varphi_A(x), t+1).$$

Then it is shown by Jorgensen and Thurston that M_A admits a complete hyperbolic structure. Let $\rho : \pi_1(\boldsymbol{T}) \to PSL(2,\mathbb{C})$ be the restriction of the holonomy representation of $\pi_1(M_A)$ to the subgroup $\pi_1(\boldsymbol{T})$. Then we have $(\nu_-(\rho), \nu_+(\rho)) = (\mu_-, \mu_+)$, where μ_+ and μ_-, respectively, are the slopes of the attractive and repulsive eigen spaces of A. This can be see as follows. Consider the infinite cyclic cover $\tilde{M}_A = \boldsymbol{T} \times \mathbb{R}$ of the complete hyperbolic manifold M_A. Then the covering tansformation $(x,t) \mapsto (\varphi_A(x), t+1)$ determines a hyperbolic isometry, h, of $\tilde{M}_A$. Now pick any essential simple loop γ in $\boldsymbol{T}$, and consider its geodesic representative γ^* in $\tilde{M}_A$. Then the closed geodesics $h^n(\gamma^*)$ are eventually contained in any neighborhood of e_+ as $n \to \infty$. Since the slope of the simple loops $h^n(\gamma)$ converges to μ_+, this implies that $\nu_+(\rho) = \mu_+$. Similarly, we have $\nu_-(\rho) = \mu_-$.

Remark 2.2. In the definition of the end invariant of a geometrically infinite end, the loops γ_n can be chosen so that the length $\ell(\gamma_n^*)$ is bounded above by a constant. This is because, we can extend each γ_n^* to a pleated surface, and we can find a simple loop on the pleated surface whose length is bounded above by some constant (see [2, 16]).

If both two ends $\nu_-(\rho)$ and $\nu_+(\rho)$ lie in the Teichmüller space, then the group Γ is *quasi-Fuchsian*, namely, there is a self-homeomorphism $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which conjugates the representation ρ to a *Fuchsian* representation $\rho_0 : \pi_1(\boldsymbol{T}) \to PSL(2,\mathbb{R})$, i.e.,

$$\rho(g) = Q \circ \rho_0(g) \circ Q^{-1}$$

for all $g \in \pi_1(\boldsymbol{T})$. For quasi-Fuchsian representations, the pair of the end invariants $(\nu_-(\rho), \nu_+(\rho))$ completely determines the group. To be precise, let $\mathcal{QF}(\boldsymbol{T})$ be the space of quasi-Fuchsian representations of $\pi_1(\boldsymbol{T})$. Then the following is due to Bers.

Theorem 2.3. *The space $\mathcal{QF}(\boldsymbol{T})$ is homeomorphic to $\mathcal{T}(\boldsymbol{T}) \times \mathcal{T}(\boldsymbol{T}) \cong \mathbb{H}^2 \times \mathbb{H}^2$ via the correspondence*

$$\rho \leftrightarrow (\nu_-(\rho), \nu_+(\rho)) = (\Omega_-/\Gamma, \Omega_+/\Gamma).$$

Let $\mathcal{D}(\boldsymbol{T})$ be the space of discrete faithful type-preserving representations of $\pi_1(\boldsymbol{T})$, modulo conjugation by elements of $PSL(2,\mathbb{C})$. Minsky [11] established the following theorem which solves the *density conjecture* and the *ending lamination conjecture* of Thurston for punctured torus groups.

Theorem 2.4. (1) *$\mathcal{D}(\boldsymbol{T})$ is equal to the closure (in the representation space) of $\mathcal{QF}(\boldsymbol{T})$.*

(2) *The map $\rho \mapsto (\nu_-(\rho), \nu_+(\rho))$ determines a bijective correspondence between $\mathcal{D}(\boldsymbol{T})$ and $\mathbb{H}^2 \times \mathbb{H}^2 - \mathrm{diag}(\partial\mathbb{H}^2)$.*

Remark 2.5. It is also shown that the map $\rho \mapsto (\nu_-(\rho), \nu_+(\rho))$ is not continuos, whereas its inverse map is continuous.

3. Bowditch, Tan-Wong-Zhang end invariants

Motivated by the definition of the end of a geometrically infinite of a Kleinian group, Bowditch [3] introduced the notion of the end invariants of an arbitrary type-preserving $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. Tan, Wong and Zhang [14, 15] extended this notion (with slight modification) to an arbitrary $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. To describe this, let $\mathcal{C}$ be the set of free homotopy classes of essential simple loops on $\boldsymbol{T}$. Then $\mathcal{C}$ is identified with $\hat{\mathbb{Q}}$, the vertex set of the Farey tessellation $\mathcal{D}$, by the following rule $s \mapsto \beta_s$, where β_s is the image of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope s in $\boldsymbol{T} = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$. The projective lamination space $\mathcal{PL}$ of $\boldsymbol{T}$ is then identified with $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and contains $\mathcal{C}$ as the dense subset of rational points.

Definition 3.1. Let ρ be a PSL$(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$.

(1) An element $X \in \mathcal{PL}$ is an *end invariant* of ρ if there exists a sequence of distinct elements $X_n \in \mathcal{C}$ such that $X_n \to X$ and that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above.

(2) $\mathcal{E}(\rho)$ denotes the set of end invariants of ρ.

In the above definition, it should be noted that $|\mathrm{tr}\rho(X_n)|$ is well-defined though $\mathrm{tr}\rho(X_n)$ is defined only up to sign. Note also that the condition that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above is equivalent to the condition that the (real) hyperbolic translation lengths of the isometries $\rho(X_n)$ of $\mathbb{H}^3$ are bounded from above. So, if ρ is a faithful discrete type-preserving representation and ν is the end invariant of a geometrically infinite end of the quotient hyperbolic manifold, then ν is an end invariant of ρ in the sense of Definition 3.1 by virtue of Remark 2.2.

Tan, Wong and Zhang [14, 15] showed that $\mathcal{E}(\rho)$ is a closed subset of $\mathcal{PL}$ and proved various interesting properties of $\mathcal{E}(\rho)$, including a characterization of those representations ρ with $\mathcal{E}(\rho) = \emptyset$ or $\mathcal{PL}$, generalizing results of Bowditch [3]. They also proposed an interesting conjecture [15, Conjecture 1.8] concerning possible homeomorphism types of $\mathcal{E}(\rho)$. The following is a modified version of the conjecture which Tan [13] informed to the authors.

Conjecture 3.2. Suppose $\mathcal{E}(\rho)$ has at least two accumulation points. Then either $\mathcal{E}(\rho) = \mathcal{PL}$ or a Cantor set of $\mathcal{PL}$.

They constructed a family of representations ρ which have Cantor sets as $\mathcal{E}(\rho)$, and proved the following supporting evidence to the conjecture (see [15, Theorem 1.7]).

Theorem 3.3. *Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2,\mathbb{C})$ be* discrete *in the sense that the set $\{\mathrm{tr}(\rho(X)) \mid X \in \mathcal{C}\}$ is discrete in $\mathbb{C}$. Then if $\mathcal{E}(\rho)$ has at least three elements, then $\mathcal{E}(\rho)$ is either a Cantor set of $\mathcal{PL}$ or all of $\mathcal{PL}$.*

However, the above set does not describe the set $\mathcal{E}(\rho)$ explicitly. In the next section, we give an infinite family of representations ρ for which $\mathcal{E}(\rho)$ is an explicitly described Cantor set (Theorem 4.1).

4. The set of end invariants of the holonomy representation of a hyperbolic 2-bridge link

Consider the discrete group, H, of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Let $\boldsymbol{S} := (\mathbb{R}^2 - \mathbb{Z}^2)/H$ be the quotient 4-times punctures sphere. Let $\tilde{H}$ be the groups of transformations on $\mathbb{R}^2 - \mathbb{Z}^2$ generated by π-rotations about points in $(\frac{1}{2}\mathbb{Z})^2$, and set $\boldsymbol{O} = (\mathbb{R}^2 - \mathbb{Z}^2)/\tilde{H}$. Then $\boldsymbol{O}$ is the $(2,2,2,\infty)$-orbifold (i.e., the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle π). There is a $\mathbb{Z}_2$-covering $\boldsymbol{T} \to \boldsymbol{O}$ and a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-covering $\boldsymbol{S} \to \boldsymbol{O}$: the pair of these coverings is called the *Fricke diagram*, and each of $\boldsymbol{T}$, $\boldsymbol{S}$, and $\boldsymbol{O}$ is called a *Fricke surface* (see [12]).

A simple loop in a Fricke surface is said to be *essential* if it does not bound a disk, a disk with one puncture, or a disk with one cone point. Similarly, a simple arc in a Fricke surface joining punctures is said to be *essential* if it does not cut off a "monogon", i.e., a disk minus a point on the boundary. Then the isotopy classes of essential simple loops (essential simple arcs with one end in a given puncture, respectively) in a Fricke surface are in one-to-one correspondence with $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\}$: A representative of the isotopy class corresponding to $r \in \hat{\mathbb{Q}}$ is the projection of a line in $\mathbb{R}^2$ (the line being disjoint from $\mathbb{Z}^2$ for the loop case, and intersecting $\mathbb{Z}^2$ for the arc case). The element $r \in \hat{\mathbb{Q}}$ associated to a loop or an arc is called its *slope*. An essential simple loop of slope s in $\boldsymbol{T}$ or $\boldsymbol{O}$ is denoted by β_s, while that in $\boldsymbol{S}$ is denoted by α_s. The notation reflects the following fact: After an isotopy, the restriction of the projection $\boldsymbol{T} \to \boldsymbol{O}$ to β_s ($\subset \boldsymbol{T}$) gives a homeomorphism from β_r ($\subset \boldsymbol{T}$) to β_s ($\subset \boldsymbol{O}$), while the restriction of the projection $\boldsymbol{S} \to \boldsymbol{O}$ to α_s gives a two-fold covering from α_s ($\subset \boldsymbol{S}$) to β_s ($\subset \boldsymbol{O}$).

Now let $K(r)$ be a 2-bridge link of slope r. Then the link complement $S^3 - K(r)$ is obtained from $\boldsymbol{S} \times [-1,1]$ by adding 2-handles along the loops $\alpha_\infty \times \{-1\}$ and $\alpha_r \times \{1\}$. Hence the link group $\pi_1(S^3 - K(r))$ is identified with $\pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$. Now assume that $K(r)$ is hyperbolic. Let ρ_r be the $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{S})$ obtained as the composition

$$\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(S^3 - K(r)) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C}),$$

where the last homomorphism is the holonomy representation of the complete hyperbolic structure of $S^3 - K(r)$. Since $S^3 - K(r)$ is generated by two meridians, $\rho_r(\pi_1(\boldsymbol{S}))$ is generated by two parabolic transformations. Hence the hyperbolic manifold $S^3 - K(r)$ admits an isometric $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action (see [16, Section 5.4] and Figure knot-symmetry) and so the $PSL(2,\mathbb{C})$-representation ρ_r of $\pi_1(\boldsymbol{S})$ extends to that of $\pi_1(\boldsymbol{O})$. Moreover, this extension is unique (see [1, Proposition 2.2]). So we obtain, in a unique way, a $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$ by restriction. We continue to denote it by ρ_r. Our main result gives an explicit description of the set $\mathcal{E}(\rho_r)$.

To state the main result, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r, and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. Then the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbb{H}^2$ (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R. Suppose that r is a rational number with

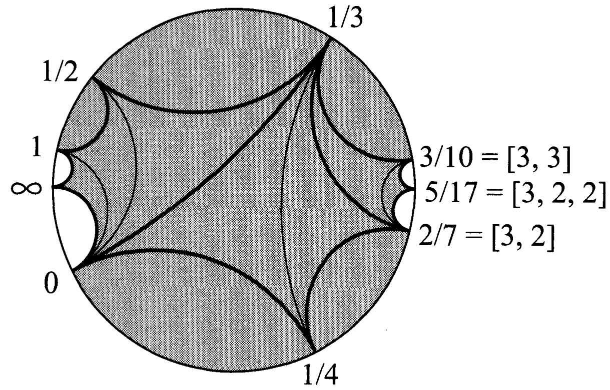

FIGURE 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

$0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write

$$r = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} =: [a_1, a_2, \dots, a_n],$$

where $n \geq 1$, $(a_1, \dots, a_n) \in (\mathbb{Z}_+)^n$, and $a_n \geq 2$. Then the above intervals are given by $I_1(r) = [0, r_1]$ and $I_2(r) = [r_2, 1]$, where

$$r_1 = \begin{cases} [a_1, a_2, \dots, a_{n-1}] & \text{if } n \text{ is odd,} \\ [a_1, a_2, \dots, a_{n-1}, a_n - 1] & \text{if } n \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [a_1, a_2, \dots, a_{n-1}, a_n - 1] & \text{if } n \text{ is odd,} \\ [a_1, a_2, \dots, a_{n-1}] & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4.1. *For a hyperbolic 2-bridge link $K(r)$, the set $\mathcal{E}(\rho_r)$ is equal to the limit set $\Lambda(\hat{\Gamma}_r)$ of the group $\hat{\Gamma}_r$.*

The proof is based on (1) the (well-known) discreteness of the marked length spectrum of the (geometrically finite) hyperbolic manifold $S^3 - K(r)$, (2) Bowditch's result [3, Proposition 3.13] on the end invariants, and (3) complete answers, obtained in the series of papers [4, 5, 6, 7] (see also the announcement [8]), to the following question concerning the simple loops in 2-bridge sphere $\boldsymbol{S}$ of a 2-bridge link $K(r)$.

(1) Which simple loop on $\boldsymbol{S}$ is null-homotopic or pheripehral on $S^3 - K(r)$?
(2) For given two simple loops on $\boldsymbol{S}$, when are they homotopic?

For the details of the proof, please see [9].

At the end of this note, we would like to propose the following conjecture.

Conjecture 4.2. Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2, \mathbb{C})$ be a type-preserving representation such that $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_r)$. Then ρ is conjugate to the representation ρ_r.

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